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Edge-Betweenness Centrality Measure of Wheel-related Graphs

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Abstract

Edge-betweenness centrality is a key metric in network analysis that assigns a real value to each edge in a graph, reflecting its importance in terms of information flow or connectivity. It is particularly useful in the study of biological and social networks for identifying critical connections. In this article, we investigate the edge-betweenness centrality of edges in several wheel-related graphs, including the helm graph, gear graph, closed flower graph, triangulated wheel graph, and sunflower graph. Our results provide insights into the structural properties of these graphs and contribute to the broader understanding of edge significance in complex networks.

Keywords: double wheel; helm graph; gear graph; closed flower graph; sunflower graph; quality education (SDG).

1 Introduction

We considered simple, finite, undirected, and connected graphs with the vertex set $V(G)$ and edge set $E(G)$. If two vertices u and v are adjacent, we write $u \sim v$ and the edge between u and v is denoted by uv. The *distance* between two vertices u and v in a graph G, denoted by $d_G(u,v)$, is defined as the length of the shortest $u-v$ path in G. In a $u-v$ path given by $P(u,v) = (u = w_1, w_2, \dots, w_k = v)$, the edges $w_iw_{i+1},\ 1 \leq i \leq k-1$ are known as the internal edges of the path $P(u,v)$. For standard terminology and notions in graph theory, we follow the text book by [4].

In any given graph, a vertex or an edge plays a significant role. The centrality measure is one parameter that is generally used to study the importance of a vertex/edge in a network. It is a function that assigns a real value to the vertices/edges of a graph. Most of these centrality measures are defined on the vertex set of the graph. One such important vertex centrality measure is the stress centrality [10] of a vertex v in a graph G which is defined as the number of shortest paths that pass through v. The stress centrality of vertices in the splitting graph, the shadow graph, and the Mycielski graph of a graph are discussed in [7].

Topological indices are important because they quantitatively relate a molecule's structure to its physical, chemical, and biological properties, enabling prediction of behavior without experimental testing. The general formula for various Zagreb indices of prime coprime graphs have been studied in [1], where as the Sombor index of the power graph for some finite non-abelian groups are determined in [2]. Topological indices related to stress centrality is studied in [9]. The authors of the article [6] explored the properties of stress sum matrix which is based on stress sum index. However, one can observe that very little attention is given to the edges, though they play an important role in connectivity and information flow. Edge-betweenness centrality measure (EBCM) is a centrality measure that focus on the edges in a graph. The concept of EBCM was introduced in [3], where the author called it the "rush".

Definition 1.1. (*Edge-Betweenness Centrality Measure*). *Given a graph G and an edge $e \in E(G)$, the EBCM of e denoted by $\mathcal{E}(e)$ is given by,*

$$\mathcal{E}(e) = \sum_{\substack{u,v \in V(G) \\ u \neq v}} \frac{\sigma_e(u,v)}{\sigma(u,v)},$$

where $\sigma(u,v)$ denotes the number of shortest $u-v$ paths in G and $\sigma_e(u,v)$ denotes the number of shortest $u-v$ paths in G that contain edge e.

In the literature, the importance of the EBCM is discussed in [5]. The expression for the stress centrality of vertices in various wheel-related graphs has been obtained in [8]. The main focus of this article is the expressions for the EBCM of edges in various wheel-related graphs, which is discussed in the next section.

2 Main Results

In this section, we obtain EBCM of all the edges in a few wheel-related graphs. Let $G = C_n$ be a cycle graph with $V(G) = \{w_1, w_2, \dots, w_n\}$ and $w_i \sim w_{i+1}, w_1 \sim w_n, 1 \leq i \leq n-1$. Then $W_{n+1} = G + w$ is called a wheel graph. The vertex w is called the central vertex of W_{n+1}.

Theorem 2.1. *Let $G = DW_n = 2C_n + K_1$ be a double wheel graph on $2n+1$ vertices. Let $V(G) = \{w_1, \ldots, w_n, w_1', \ldots, w_n', w\}$, where $w_i, w_i', 1 \leq i \leq n$ are the vertices of the cycle graph $2C_n$ and w be the central vertex. Then,*

$$\mathcal{E}(e) = \begin{cases} 2n-3, & e = ww_i, ww_i', 1 \leq i \leq n, \\ 2, & e = w_iw_{i+1}, w_i'w_{i+1}', w_1w_n, w_1'w_n', 1 \leq i \leq n-1. \end{cases}$$

Proof. Let $e = ww_1$. As the diameter of G is two,

$$\begin{aligned} \mathcal{E}(e) &= \frac{\sigma_e(w, w_1)}{\sigma(w, w_1)} + \sum_{j=1}^{n} \frac{\sigma_e(w_1, w_j')}{\sigma(w_1, w_j')} + \sum_{j=3,n-1} \frac{\sigma_e(w_1, w_j)}{\sigma(w_1, w_j)} + \sum_{j=4}^{n-2} \frac{\sigma_e(w_1, w_j)}{\sigma(w_1, w_j)} \\ &= 1 + \sum_{j=1}^{n} 1 + 2\left(\frac{1}{2}\right) + \sum_{j=4}^{n-2} 1 \\ &= 1 + n + 1 + n - 5 = 2n - 3. \end{aligned}$$

Due to symmetry, $\mathcal{E}(ww_i) = \mathcal{E}(ww_i') = 2n-3, 1 \leq i \leq n$.

Let $e = w_1w_2$. Then,

$$\begin{aligned} \sigma_e(w_n, w_2) = 1, \sigma(w_n, w_2) = 2, \\ \sigma_e(w_1, w_3) = 1, \sigma(w_1, w_3) = 2, \\ \sigma_e(w_1, w_2) = \sigma(w_1, w_2) = 1. \end{aligned}$$

Now from symmetry,

$$\mathcal{E}(w_iw_{i+1}) = \mathcal{E}(w_i'w_{i+1}') = 2, 1 \leq i \leq n-1,$$

and $\mathcal{E}(w_1w_n) = \mathcal{E}(w_1'w_n') = 2$. □

Definition 2.1. *The gear graph G_{n+1} is formed from the wheel graph W_{n+1} by subdividing each rim edge $w_iw_{i+1}, 1 \leq i \leq n-1$ and w_nw_1 of the cycle C_n by inserting a new vertex $u_j, 1 \leq j \leq n$. Note that the diameter of G_{n+1} is 4.*

Theorem 2.2. *Let $G = G_{n+1}$ and $V(G) = \{w, w_1, \ldots, w_n, u_1, \ldots, u_n\}$. Then,*

$$\mathcal{E}(e) = \begin{cases} 4, & e \in E(G), \textit{ for } n = 3 \\ \frac{22}{3}, & e = ww_i, 1 \leq i \leq 4 \textit{ for } n = 4 \\ \frac{16}{3}, & e = w_iu_i, u_jw_{j+1}, u_4w_1, 1 \leq i \leq 4, 1 \leq j \leq 3, \textit{ for } n = 4 \\ \frac{60n-131}{15}, & e = ww_i, 1 \leq i \leq n \textit{ for } n \geq 5 \\ \frac{30n+41}{30}, & e = w_iu_i, u_jw_{j+1}, u_nw_1, 1 \leq i \leq 4, 1 \leq j \leq n-1, \textit{ for } n \geq 5 \end{cases}.$$

Proof. Case 1: Let $n = 3$. Let $e = ww_1$. Then,

$$\begin{aligned}\mathcal{E}(e) &= \frac{\sigma_e(w,w_1)}{\sigma(w,w_1)} + \frac{\sigma_e(w,u_1)}{\sigma(w,u_1)} + \frac{\sigma_e(w,u_3)}{\sigma(w,u_3)} + \sum_{j=2}^{3}\frac{\sigma_e(w_1,w_j)}{\sigma(w_1,w_j)} + \frac{\sigma_e(w_1,u_2)}{\sigma(w_1,u_2)}\\ &\quad + \frac{\sigma_e(u_1,w_3)}{\sigma(u_1,w_3)} + \frac{\sigma_e(u_3,w_2)}{\sigma(u_3,w_2)}\\ &= 1 + \frac{1}{2} + \frac{1}{2} + 2\times\frac{1}{2} + \frac{2}{4} + \frac{1}{4} + \frac{1}{4} = 4.\end{aligned}$$

Due to symmetry, $\mathcal{E}(ww_i) = 4, 1 \leq i \leq 3$.

Let $e = w_1u_1$. Then,

$$\begin{aligned}\mathcal{E}(e) &= \frac{\sigma_e(w_1,u_1)}{\sigma(w_1,u_1)} + \frac{\sigma_e(w_1,w_2)}{\sigma(w_1,w_2)} + \frac{\sigma_e(w_1,u_2)}{\sigma(w_1,u_2)} + \frac{\sigma_e(u_1,u_3)}{\sigma(u_1,u_3)} + \frac{\sigma_e(u_1,w_3)}{\sigma(u_1,w_3)}\\ &\quad + \frac{\sigma_e(u_3,w_2)}{\sigma(u_3,w_2)} + \frac{\sigma_e(u_1,w)}{\sigma(u_1,w)}\\ &= 1 + \frac{1}{2} + \frac{1}{4} + 1 + \frac{2}{4} + \frac{1}{4} + \frac{1}{2} = 4.\end{aligned}$$

Due to symmetry, $\mathcal{E}(u_iw_i) = \mathcal{E}(u_jw_{j+1}) = \mathcal{E}(u_3w_1) = 4, 1 \leq i \leq 3, 1 \leq j \leq 2$.

Case 2: Let $n = 4$. Let $e = ww_1$. Then,

$$\begin{aligned}\mathcal{E}(e) &= \frac{\sigma_e(w,w_1)}{\sigma(w,w_1)} + \frac{\sigma_e(w_1,w_3)}{\sigma(w_1,w_3)} + \frac{\sigma_e(w_1,w_2)}{\sigma(w_1,w_2)} + \frac{\sigma_e(w_1,u_3)}{\sigma(w_1,u_3)} + \frac{\sigma_e(w_1,u_2)}{\sigma(w_1,u_2)}\\ &\quad + \frac{\sigma_e(w_1,w_4)}{\sigma(w_1,w_4)} + \frac{\sigma_e(w,u_1)}{\sigma(w,u_1)} + \frac{\sigma_e(w,u_4)}{\sigma(w,u_4)} + \frac{\sigma_e(w_3,u_1)}{\sigma(w_3,u_1)} + \frac{\sigma_e(w_3,u_4)}{\sigma(w_3,u_4)}\\ &\quad + \frac{\sigma_e(w_2,u_4)}{\sigma(w_2,u_4)} + \frac{\sigma_e(w_4,u_1)}{\sigma(w_4,u_1)} + \frac{\sigma_e(u_2,u_4)}{\sigma(u_2,u_4)} + \frac{\sigma_e(u_1,u_3)}{\sigma(u_1,u_3)}\\ &= 1 + 1 + \frac{1}{2} + \frac{2}{3} + \frac{2}{3} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{2}{6} + \frac{2}{6} = \frac{22}{3}.\end{aligned}$$

Due to symmetry, $\mathcal{E}(ww_i) = \dfrac{22}{3}, 1 \leq i \leq 4$.

Let $e = w_1u_1$. Then,

$$\begin{aligned}\mathcal{E}(e) &= \frac{\sigma_e(w_1,u_1)}{\sigma(w_1,u_1)} + \frac{\sigma_e(w_1,w_2)}{\sigma(w_1,w_2)} + \frac{\sigma_e(w,u_1)}{\sigma(w,u_1)} + \frac{\sigma_e(u_1,u_4)}{\sigma(u_1,u_4)} + \frac{\sigma_e(w_1,u_2)}{\sigma(w_1,u_2)}\\ &\quad + \frac{\sigma_e(w_3,u_1)}{\sigma(w_3,u_1)} + \frac{\sigma_e(w_4,u_1)}{\sigma(w_4,u_1)} + \frac{\sigma_e(u_1,u_3)}{\sigma(u_1,u_3)} + \frac{\sigma_e(u_2,u_4)}{\sigma(u_2,u_4)} + \frac{\sigma_e(w_2,u_4)}{\sigma(w_2,u_4)}\\ &= 1 + \frac{1}{2} + \frac{1}{2} + 1 + \frac{1}{3} + \frac{1}{3} + \frac{2}{3} + \frac{3}{6} + \frac{1}{6} + \frac{1}{3} = \frac{16}{3}.\end{aligned}$$

Due to symmetry, $\mathcal{E}(u_iw_i) = \mathcal{E}(u_jw_{j+1}) = \mathcal{E}(u_4w_1) = \dfrac{16}{3}, 1 \leq i \leq 4, 1 \leq j \leq 3$.

Case 3: Let $n \geq 5$. Let $e = ww_1$. The shortest $x - y$ paths with e as an internal edge are listed below.

Shortest path	$\sigma_e(x,y)$	$\sigma(x,y)$	$\mathcal{E}(e)$
$w - w_1$	1	1	1
$w - u_j, j = 1, n$	1	2	$\frac{1}{2} \times 2$
$w_1 - w_j, 3 \leq j \leq n-1$	1	1	$n-3$
$w_1 - w_j, j = 2, n$	1	2	$\frac{1}{2} \times 2$
$w_1 - u_j, 3 \leq j \leq n-2$	2	2	$n-4$
$w_1 - u_j, j = 2, n-1$	2	3	$\frac{2}{3} \times 2$
$u_1 - u_j, j = 3, n-1$	2	5	$\frac{2}{5} \times 2$
$u_n - u_j, j = 2, n-2$	2	5	$\frac{2}{5} \times 2$
$u_n - w_j, j = 2, n-1$	1	3	$\frac{1}{3} \times 2$
$u_1 - w_j, j = 3, n$	1	3	$\frac{1}{3} \times 2$
$u_1 - w_j, 4 \leq j \leq n-1$	1	2	$\frac{1}{2} \times (n-4)$
$u_n - w_j, 3 \leq j \leq n-2$	1	2	$\frac{1}{2} \times (n-4)$
$u_1 - u_j, 4 \leq j \leq n-2$	2	4	$\frac{2}{4} \times (n-5)$
$u_n - u_j, 3 \leq j \leq n-3$	2	4	$\frac{2}{4} \times (n-5)$

Hence,

$$\mathcal{E}(ww_i) = \frac{60n - 131}{15}, 1 \leq i \leq n.$$

Let $e = w_1 u_1$.

Shortest path	$\sigma_e(x,y)$	$\sigma(x,y)$	$\mathcal{E}(e)$
$w_1 - u_1, u_1 - u_n$	1	1	2
$u_n - w_2, u_1 - w_3, w_1 - u_2$	1	3	$\frac{1}{3} \times 3$
$u_1 - w_j, 4 \leq j \leq n-1$	1	2	$\frac{1}{2} \times (n-4)$
$u_2 - u_n$	1	5	$\frac{1}{5}$
$u_1 - u_3$	2	5	$\frac{2}{5}$
$u_1 - u_{n-1}$	3	5	$\frac{3}{5}$
$u_1 - w_n$	2	3	$\frac{2}{3}$
$u_1 - u_j, 4 \leq j \leq n-2$	2	4	$\frac{2}{4} \times (n-5)$
$w_1 - w_2, w - u_1$	1	2	$2 \times \frac{1}{2}$

Hence,

$$\mathcal{E}(w_n u_n) = \mathcal{E}(u_n w_1) = \mathcal{E}(w_i u_i) = \mathcal{E}(u_i w_{i+1}) = \frac{30n + 41}{30}, 1 \leq i \leq n-1.$$

□

Definition 2.2. *The helm graph is denoted by H_{n+1} is formed from the wheel graph W_{n+1} by adding, for each rim vertex $w_i, 1 \leq i \leq n$, a pendant vertex u_i that is adjacent to w_i. The diameter of H_{n+1} is four.*

Theorem 2.3. *Let $G = H_{n+1}$, and $V(G) = \{w, w_1, \ldots, w_n, u_1, \ldots, u_n\}$. Then,*

$$\mathcal{E}(e) = \begin{cases} 2 & ; e = ww_i, 1 \leq i \leq 3 \text{ for } n = 3 \\ 4 & ; e = w_1w_2, w_2w_3, w_3w_1, \text{ for } n = 3 \\ \frac{10}{3} & ; e = ww_i, 1 \leq i \leq 4, \text{ for } n = 4 \\ \frac{20}{3} & ; e = w_iw_{i+1}, w_1w_4, 1 \leq i \leq 3, \text{ for } n = 4 \\ 4n - 14 & ; e = ww_i, 1 \leq i \leq n \text{ for } n \geq 5 \\ 8 & ; e = w_iw_{i+1}, w_1w_n, 1 \leq i \leq n-1 \text{ for } n \geq 5 \\ 2n & ; e = w_iu_i, 1 \leq i \leq n \text{ for } n \geq 3 \end{cases}.$$

Proof. Case 1: Let $n = 3$. Let $e = ww_1$. Then, $\sigma_e(w, w_1) = \sigma(w, w_1) = \sigma_e(w, u_1) = \sigma(w, u_1) = 1$. Due to symmetry, $\mathcal{E}(ww_i) = 2, 1 \leq i \leq 3$.

Let $e = w_1w_2$. Then,

$$\begin{aligned} \mathcal{E}(e) &= \frac{\sigma_e(w_1, w_2)}{\sigma(w_1, w_2)} + \frac{\sigma_e(w_1, u_2)}{\sigma(w_1, u_2)} + \frac{\sigma_e(u_1, w_2)}{\sigma(u_1, w_2)} + \frac{\sigma_e(u_1, u_2)}{\sigma(u_1, u_2)} \\ &= 1 + 1 + 1 + 1 = 4. \end{aligned}$$

Due to symmetry, $\mathcal{E}(w_1w_2) = \mathcal{E}(w_2w_3) = \mathcal{E}(w_3w_1) = 4$.

Case 2: Let $n = 4$. Let $e = ww_1$. Then,

$$\begin{aligned} \mathcal{E}(e) &= \frac{\sigma_e(w, w_1)}{\sigma(w, w_1)} + \frac{\sigma_e(w, u_1)}{\sigma(w, u_1)} + \frac{\sigma_e(w_1, w_3)}{\sigma(w_1, w_3)} + \frac{\sigma_e(u_1, w_3)}{\sigma(u_1, w_3)} + \frac{\sigma_e(u_1, u_3)}{\sigma(u_1, u_3)} + \frac{\sigma_e(u_3, w_1)}{\sigma(u_3, w_1)} \\ &= 1 + 1 + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{10}{3}. \end{aligned}$$

Due to symmetry, $\mathcal{E}(ww_i) = \dfrac{10}{3}, 1 \leq i \leq 4$.

Let $e = w_1w_2$. Then,

$$\begin{aligned} \mathcal{E}(e) &= \frac{\sigma_e(w_1, w_2)}{\sigma(w_1, w_2)} + \frac{\sigma_e(w_1, u_2)}{\sigma(w_1, u_2)} + \frac{\sigma_e(u_1, w_2)}{\sigma(u_1, w_2)} + \frac{\sigma_e(u_1, u_2)}{\sigma(u_1, u_2)} + \frac{\sigma_e(w_1, w_3)}{\sigma(w_1, w_3)} + \frac{\sigma_e(w_1, u_3)}{\sigma(w_1, u_3)} \\ &\quad + \frac{\sigma_e(u_1, w_3)}{\sigma(u_1, w_3)} + \frac{\sigma_e(u_1, u_3)}{\sigma(u_1, u_3)} + \frac{\sigma_e(w_2, w_4)}{\sigma(w_2, w_4)} + \frac{\sigma_e(w_2, u_4)}{\sigma(w_2, u_4)} + \frac{\sigma_e(u_2, w_4)}{\sigma(u_2, w_4)} + \frac{\sigma_e(u_2, u_4)}{\sigma(u_2, u_4)} \\ &= 1 + 1 + 1 + 1 + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{20}{3}. \end{aligned}$$

Due to symmetry, $\mathcal{E}(w_iw_{i+1}) = \mathcal{E}(w_4w_1) = \dfrac{20}{3}, 1 \leq i \leq 3$.

Case 3: Let $n \geq 5$. Let $e = ww_1$.

Shortest path	$\sigma_e(x, y)$	$\sigma(x, y)$	$\mathcal{E}(e)$
$w - w_1, u_1 - w$	1	1	2
$w_1 - w_j, w_1 - u_j, j = 3, n-1$	1	2	$\frac{1}{2} \times 4$
$u_1 - w_j, w_1 - u_j, j = 3, n-1$	1	2	$\frac{1}{2} \times 4$
$u_1 - w_j, u_1 - u_j, w_1 - w_j, w_1 - u_j, 4 \leq j \leq n-2$	1	1	$4(n-5)$

Hence, $\mathcal{E}(ww_i) = 4n - 14, 1 \leq i \leq n$.

Let $e = w_1w_2$.

Shortest path	$\sigma_e(x,y)$	$\sigma(x,y)$	$\mathcal{E}(e)$
$w_1 - w_3, w_1 - u_3, u_n - w_2, u_n - u_2$	1	2	$\frac{1}{2} \times 4$
$w_1 - u_2, w_1 - w_2, u_1 - u_2, u_1 - w_2$	1	1	4
$w_n - w_2, w_n - u_2, u_1 - w_3, u_1 - u_3$	1	2	$\frac{1}{2} \times 4$

Hence, $\mathcal{E}(w_iw_{i+1}) = \mathcal{E}(w_1w_n) = 8, 1 \leq i \leq n-1$.

Case 4: Let $n \geq 3$. Let $e = w_1u_1$.

$$\begin{aligned}\mathcal{E}(e) &= \frac{\sigma_e(u_1,w)}{\sigma(u_1,w)} + \frac{\sigma_e(u_1,w_1)}{\sigma(u_1,w_1)} + \sum_{j=2}^{n}\frac{\sigma_e(u_1,w_j)}{\sigma(u_1,w_j)} + \sum_{j=2}^{n}\frac{\sigma_e(u_1,u_j)}{\sigma(u_1,u_j)}\\ &= 1 + 1 + \sum_{j=2}^{n} 1 + \sum_{j=2}^{n} 1\\ &= 1 + 1 + n - 1 + n - 1 = 2n.\end{aligned}$$

Hence, $\mathcal{E}(w_iu_i) = 2n, 1 \leq i \leq n$. $\square$

Definition 2.3. *The flower graph is denoted by Fl_{n+1} is formed from the wheel graph W_{n+1}, by adding the vertices $\{u_1, u_2, \ldots, u_n\}$ where each $u_i, 1 \leq i \leq n$ is connected to the corresponding rim vertices $w_i, 1 \leq i \leq n$. Additionally, each $u_i, 1 \leq i \leq n$ is also connected to the central vertex w. Note that the diameter of Fl_{n+1} is 2.*

Theorem 2.4. *Let $G = Fl_{n+1}, V(G) = \{w, w_1, \ldots, w_n, u_1, \ldots, u_n\}$. Then,*

$$\mathcal{E}(e) = \begin{cases} 2 & ; e = ww_i, 1 \leq i \leq 3, \textit{ for } n = 3\\ 2 & ; e = w_1w_2, w_2w_3, w_3w_1, \textit{ for } n = 3\\ \frac{10}{3} & ; e = ww_i, 1 \leq i \leq 4, \textit{ for } n = 4\\ \frac{8}{3} & ; e = w_iw_{i+1}, w_1w_4, 1 \leq i \leq 3, \textit{ for } n = 4\\ 2n-5 & ; e = ww_i; 1 \leq i \leq n, \textit{ for } n \geq 5\\ 3 & ; e = w_iw_{i+1}, w_1w_n, 1 \leq i \leq n-1, \textit{ for } n \geq 5\\ 2 & ; e = w_iu_i, 1 \leq i \leq n, \textit{ for } n \geq 3\\ 2n-2 & ; e = wu_i, 1 \leq i \leq n, \textit{ for } n \geq 3 \end{cases}.$$

Proof. Case 1: Let $n = 3$. Let $e = ww_1$. Then,

$$\mathcal{E}(e) = \frac{\sigma_e(w,w_1)}{\sigma(w,w_1)} + \frac{\sigma_e(w_1,u_2)}{\sigma(w_1,u_2)} + \frac{\sigma_e(w_1,u_3)}{\sigma(w_1,u_3)} = 1 + \frac{1}{2} + \frac{1}{2} = 2.$$

Due to symmetry, $\mathcal{E}(ww_i) = 2, 1 \leq i \leq 3$.

Let $e = w_1w_2$. Then,

$$\mathcal{E}(e) = \frac{\sigma_e(w_1,w_2)}{\sigma(w_1,w_2)} + \frac{\sigma_e(w_2,u_1)}{\sigma(w_2,u_1)} + \frac{\sigma_e(w_1,u_2)}{\sigma(w_1,u_2)} = 1 + \frac{1}{2} + \frac{1}{2} = 2.$$

Due to symmetry, $\mathcal{E}(w_1w_2) = \mathcal{E}(w_2w_3) = \mathcal{E}(w_3w_1) = 2$.

Case 2: Let $n = 4$. Let $e = ww_1$. Then,

$$\begin{aligned}\mathcal{E}(e) &= \frac{\sigma_e(w,w_1)}{\sigma(w,w_1)} + \frac{\sigma_e(w_1,w_3)}{\sigma(w_1,w_3)} + \frac{\sigma_e(w_1,u_2)}{\sigma(w_1,u_2)} + \frac{\sigma_e(w_1,u_3)}{\sigma(w_1,u_3)} + \frac{\sigma_e(w_1,u_4)}{\sigma(w_1,u_4)}\\ &= 1 + \frac{1}{3} + \frac{1}{2} + 1 + \frac{1}{2} = \frac{10}{3}.\end{aligned}$$

Due to symmetry, $\mathcal{E}(ww_i) = \dfrac{10}{3}, 1 \leq i \leq 4$.

Let $e = w_1w_2$. Then,

$$\begin{aligned}\mathcal{E}(e) &= \frac{\sigma_e(w_1,w_2)}{\sigma(w_1,w_2)} + \frac{\sigma_e(w_1,w_3)}{\sigma(w_1,w_3)} + \frac{\sigma_e(w_2,w_4)}{\sigma(w_2,w_4)} + \frac{\sigma_e(w_1,u_2)}{\sigma(w_1,u_2)} + \frac{\sigma_e(w_2,u_1)}{\sigma(w_2,u_1)}\\ &= 1 + \frac{1}{3} + \frac{1}{3} + \frac{1}{2} + \frac{1}{2} = \frac{8}{3}.\end{aligned}$$

Due to symmetry, $\mathcal{E}(w_1w_2) = \mathcal{E}(w_2w_3) = \mathcal{E}(w_3w_4) = \mathcal{E}(w_1w_4) = \dfrac{8}{3}$.

Case 3: Let $n \geq 5$. Let $e = ww_1$.

Shortest path	$\sigma_e(x,y)$	$\sigma(x,y)$	$\mathcal{E}(e)$
$w - w_1$	1	1	1
$w_1 - w_j, j = 3, n-1$	1	2	$\frac{1}{2} \times 2$
$w_1 - w_j, 4 \leq j \leq n-2$	1	1	$n-5$
$w_1 - u_j, j = 2, n$	1	2	$\frac{1}{2} \times 2$
$w_1 - u_j, 3 \leq j \leq n-1$	1	1	$n-3$

Hence, $\mathcal{E}(ww_i) = 2n - 5, 1 \leq i \leq n$.

Let $e = w_1w_2$.

Shortest path	$\sigma_e(x,y)$	$\sigma(x,y)$	$\mathcal{E}(e)$
$w_1 - w_2$	1	1	1
$w_1 - u_2, w_1 - w_3, w_2 - w_n, w_2 - u_1$	1	2	$4 \times \frac{1}{2}$

Hence, $\mathcal{E}(w_iw_{i+1}) = \mathcal{E}(w_1w_n) = 3, 1 \leq i \leq n-1$.

Case 4: Let $n \geq 3$. Let $e = w_1u_1$. Then, $\sigma_e(w_1,u_1) = \sigma(w_1,u_1) = 1, \sigma_e(u_1,w_n) = \sigma_e(u_1,w_2) = 1$, $\sigma(u_1,w_n) = \sigma(u_1,w_2) = 2$. Hence, due to symmetry, $\mathcal{E}(w_iu_i) = 2, 1 \leq i \leq n$.

Let $e = wu_1$.

Shortest path	$\sigma_e(x,y)$	$\sigma(x,y)$	$\mathcal{E}(e)$
$w - u_1$	1	1	1
$u_1 - w_2, u_1 - w_n$	1	2	$2 \times \frac{1}{2}$
$u_1 - w_j, 3 \leq j \leq n-1$	1	1	$n-3$
$u_1 - u_j, 2 \leq j \leq n$	1	1	$n-1$

Hence, $\mathcal{E}(wu_i) = 2n-2, 1 \leq i \leq n$. □

Definition 2.4. *The sunflower graph, denoted as SF_{n+1}, is obtained from the wheel graph W_{n+1} by adding the set of new vertices $\{u_1, u_2, \ldots, u_n\}$, such that $u_i \sim w_i, w_{i+1}, 1 \leq i \leq n-1$ and $u_n \sim w_n, w_1$. The diameter of SF_{n+1} is 4.*

Theorem 2.5. *Let $G = SF_{n+1}$, $V(G) = \{w, w_1, \ldots, w_n, u_1, \ldots, u_n\}$. Then,*

$$\mathcal{E}(e) = \begin{cases} 2 & ; e = ww_i, 1 \leq i \leq 3, \text{ for } n = 3 \\ 2 & ; e = w_iw_{i+1}, w_1w_3, 1 \leq i \leq 2, \text{ for } n = 3 \\ \frac{7}{3} & ; e = ww_i, 1 \leq i \leq 4 \text{ for } n = 4 \\ \frac{25}{6} & ; e = w_iw_{i+1}, w_1w_4, 1 \leq i \leq 3, \text{ for } n = 4 \\ 4 & ; e = ww_i, 1 \leq i \leq 5 \text{ for } n = 5 \\ 6 & ; e = w_iw_{i+1}, w_1w_5, 1 \leq i \leq 4, \text{ for } n = 5 \\ \frac{22}{3} & ; e = ww_i, 1 \leq i \leq 6, \text{ for } n = 6 \\ \frac{20}{3} & ; e = w_iw_{i+1}, w_1w_6, 1 \leq i \leq 5, \text{ for } n = 6 \\ \frac{60n-251}{15} & ; e = ww_i; 1 \leq i \leq n, \text{ for } n \geq 7 \\ \frac{101}{15} & ; e = w_iw_{i+1}, w_1w_n, 1 \leq i \leq n-1, \text{ for } n \geq 7 \\ n & ; e = u_iw_i, u_iw_{i+1}, u_nw_n, u_nw_1, 1 \leq i \leq n-1 \text{ for } n \geq 3 \end{cases}.$$

Proof. Case 1: Let $n = 3$. Let $e = ww_1$. Then,

$$\mathcal{E}(e) = \frac{\sigma_e(w, w_1)}{\sigma(w, w_1)} + \frac{\sigma_e(w, u_1)}{\sigma(w, u_1)} + \frac{\sigma_e(w, u_3)}{\sigma(w, u_3)} = 1 + \frac{1}{2} + \frac{1}{2} = 2.$$

Due to symmetry, $\mathcal{E}(ww_i) = 2, 1 \leq i \leq 3$.

Let $e = w_1w_2$. Then,

$$\mathcal{E}(e) = \frac{\sigma_e(w_1, w_2)}{\sigma(w_1, w_2)} + \frac{\sigma_e(w_1, u_2)}{\sigma(w_1, u_2)} + \frac{\sigma_e(w_2, u_3)}{\sigma(w_2, u_3)} = 1 + \frac{1}{2} + \frac{1}{2} = 2.$$

Due to symmetry, $\mathcal{E}(w_iw_{i+1}) = \mathcal{E}(w_1w_3) = 2, 1 \leq i \leq 2$.

Case 2: Let $n = 4$. Let $e = ww_1$. Then,

$$\mathcal{E}(e) = \frac{\sigma_e(w, w_1)}{\sigma(w, w_1)} + \frac{\sigma_e(w, u_1)}{\sigma(w, u_1)} + \frac{\sigma_e(w, u_4)}{\sigma(w, u_4)} + \frac{\sigma_e(w_1, w_3)}{\sigma(w_1, w_3)} = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} = \frac{7}{3}.$$

Due to symmetry, $\mathcal{E}(ww_i) = \frac{7}{3}, 1 \leq i \leq 4$.

Let $e = w_1w_2$. Then,

$$\begin{aligned} \mathcal{E}(e) &= \frac{\sigma_e(w_1, w_2)}{\sigma(w_1, w_2)} + \frac{\sigma_e(w_1, w_3)}{\sigma(w_1, w_3)} + \frac{\sigma_e(w_2, w_4)}{\sigma(w_2, w_4)} + \frac{\sigma_e(w_1, u_2)}{\sigma(w_1, u_2)} + \frac{\sigma_e(w_2, u_4)}{\sigma(w_2, u_4)} + \frac{\sigma_e(u_2, u_4)}{\sigma(u_2, u_4)} \\ &= 1 + \frac{1}{3} + \frac{1}{3} + 1 + 1 + \frac{1}{2} = \frac{25}{6}. \end{aligned}$$

Due to symmetry, $\mathcal{E}(w_iw_{i+1}) = \mathcal{E}(w_1w_4) = \frac{25}{6}, 1 \leq i \leq 3$.

Case 3: Let $n = 5$. Let $e = ww_1$. Then,

$$\begin{aligned}\mathcal{E}(e) &= \frac{\sigma_e(w,w_1)}{\sigma(w,w_1)} + \frac{\sigma_e(w_1,w_3)}{\sigma(w_1,w_3)} + \frac{\sigma_e(w_1,w_4)}{\sigma(w_1,w_4)} + \frac{\sigma_e(w,u_1)}{\sigma(w,u_1)} + \frac{\sigma_e(w,u_5)}{\sigma(w,u_5)} \\ &\quad + \frac{\sigma_e(w_1,u_3)}{\sigma(w_1,u_3)} + \frac{\sigma_e(w_3,u_5)}{\sigma(w_3,u_5)} + \frac{\sigma_e(w_4,u_1)}{\sigma(w_4,u_1)} \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{2}{4} + \frac{1}{4} + \frac{1}{4} = 4.\end{aligned}$$

Due to symmetry, $\mathcal{E}(ww_i) = 4, 1 \leq i \leq 5$.

Let $e = w_1w_2$. Then,

$$\begin{aligned}\mathcal{E}(e) &= \frac{\sigma_e(w_1,w_2)}{\sigma(w_1,w_2)} + \frac{\sigma_e(w_1,w_3)}{\sigma(w_1,w_3)} + \frac{\sigma_e(w_1,u_2)}{\sigma(w_1,u_2)} + \frac{\sigma_e(w_2,w_5)}{\sigma(w_2,w_5)} + \frac{\sigma_e(w_2,u_5)}{\sigma(w_2,u_5)} \\ &\quad + \frac{\sigma_e(w_5,u_2)}{\sigma(w_5,u_2)} + \frac{\sigma_e(w_2,u_4)}{\sigma(w_2,u_4)} + \frac{\sigma_e(w_3,u_5)}{\sigma(w_3,u_5)} + \frac{\sigma_e(w_1,u_3)}{\sigma(w_1,u_3)} + \frac{\sigma_e(u_2,u_5)}{\sigma(u_2,u_5)} \\ &= 1 + \frac{1}{2} + 1 + \frac{1}{2} + 1 + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + 1 = 6.\end{aligned}$$

Due to symmetry, $\mathcal{E}(w_iw_{i+1}) = \mathcal{E}(w_1w_5) = 6, 1 \leq i \leq 4$.

Case 4: Let $n = 6$. Let $e = ww_1$. Then,

$$\begin{aligned}\mathcal{E}(e) &= \frac{\sigma_e(w,w_1)}{\sigma(w,w_1)} + \frac{\sigma_e(w_1,w_3)}{\sigma(w_1,w_3)} + \frac{\sigma_e(w_1,w_4)}{\sigma(w_1,w_4)} + \frac{\sigma_e(w_1,w_5)}{\sigma(w_1,w_5)} + \frac{\sigma_e(w,u_1)}{\sigma(w,u_1)} \\ &\quad + \frac{\sigma_e(w,u_6)}{\sigma(w,u_6)} + \frac{\sigma_e(w_1,u_3)}{\sigma(w_1,u_3)} + \frac{\sigma_e(w_1,u_4)}{\sigma(w_1,u_4)} + \frac{\sigma_e(u_1,u_4)}{\sigma(u_1,u_4)} + \frac{\sigma_e(u_3,u_6)}{\sigma(u_3,u_6)} \\ &\quad + \frac{\sigma_e(w_5,u_1)}{\sigma(w_5,u_1)} + \frac{\sigma_e(w_4,u_1)}{\sigma(w_4,u_1)} + \frac{\sigma_e(w_3,u_6)}{\sigma(w_3,u_6)} + \frac{\sigma_e(w_4,u_6)}{\sigma(w_4,u_6)} \\ &= 1 + \frac{1}{2} + 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{2}{3} + \frac{2}{3} + \frac{2}{6} + \frac{2}{6} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{22}{3}.\end{aligned}$$

Due to symmetry, $\mathcal{E}(ww_i) = \frac{22}{3}, 1 \leq i \leq 6$.

Let $e = w_1w_2$. Then,

$$\begin{aligned}\mathcal{E}(e) &= \frac{\sigma_e(w_1,w_2)}{\sigma(w_1,w_2)} + \frac{\sigma_e(w_1,w_3)}{\sigma(w_1,w_3)} + \frac{\sigma_e(w_2,u_6)}{\sigma(w_2,u_6)} + \frac{\sigma_e(w_1,u_2)}{\sigma(w_1,u_2)} + \frac{\sigma_e(w_2,w_6)}{\sigma(w_2,w_6)} \\ &\quad + \frac{\sigma_e(w_1,u_3)}{\sigma(w_1,u_3)} \frac{\sigma_e(u_3,u_6)}{\sigma(u_3,u_6)} + \frac{\sigma_e(u_2,u_6)}{\sigma(u_2,u_6)} + \frac{\sigma_e(w_6,u_2)}{\sigma(w_6,u_2)} + \frac{\sigma_e(w_2,u_5)}{\sigma(w_2,u_5)} \\ &\quad + \frac{\sigma_e(w_3,u_6)}{\sigma(w_3,u_6)} + \frac{\sigma_e(u_2,u_5)}{\sigma(u_2,u_5)} \\ &= 1 + \frac{1}{2} + 1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} + 1 + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{6} = \frac{20}{3}.\end{aligned}$$

Due to symmetry, $\mathcal{E}(w_iw_{i+1}) = \mathcal{E}(w_1w_6) = \frac{20}{3}, 1 \leq i \leq 5$.

Case 5: Let $n \geq 7$. Let $e = ww_1$.

Shortest path	$\sigma_e(x,y)$	$\sigma(x,y)$	$\mathcal{E}(e)$
$w - w_1$	1	1	1
$w - u_j, j = 1, n$	1	2	$\frac{1}{2} \times 2$
$w_1 - w_j, 4 \leq j \leq n-2$	1	1	$n-5$
$w_1 - w_j, j = 3, n-1$	1	2	$\frac{1}{2} \times 2$
$w_1 - u_j, 4 \leq j \leq n-3$	2	2	$n-6$
$w_1 - u_j, j = 3, n-2$	2	3	$\frac{2}{3} \times 2$
$u_1 - u_j, j = 4, n-2$	2	5	$\frac{2}{5} \times 2$
$u_n - u_j, j = 3, n-3$	2	5	$\frac{2}{5} \times 2$
$u_1 - w_j, 5 \leq j \leq n-2$	1	2	$\frac{1}{2} \times (n-6)$
$u_1 - w_j, j = 4, n-1$	1	3	$\frac{1}{3} \times 2$
$u_n - w_j, 4 \leq j \leq n-3$	1	2	$\frac{1}{2} \times (n-6)$
$u_n - w_j, j = 3, n-2$	1	3	$\frac{1}{3} \times 2$
$u_1 - u_j, 5 \leq j \leq n-3$	2	4	$\frac{2}{4} \times (n-7)$
$u_n - u_j, 4 \leq j \leq n-4$	1	2	$\frac{1}{2} \times (n-7).$

Hence, $\mathcal{E}(ww_i) = \dfrac{60n - 251}{15}, 1 \leq i \leq n.$

Let $e = w_1w_2$.

Shortest path	$\sigma_e(x,y)$	$\sigma(x,y)$	$\mathcal{E}(e)$
$w_1 - w_2, w_1 - u_2, w_2 - u_n, u_2 - u_n$	1	1	4
$w_1 - w_3, w_2 - w_n$	1	2	$\frac{1}{2} \times 2$
$w_2 - u_{n-1}, w_3 - u_n, w_1 - u_3, w_n - u_2$	1	3	$\frac{1}{3} \times 4$
$u_2 - u_{n-1}, u_3 - u_n$	1	5	$\frac{1}{5} \times 2$

Hence, $\mathcal{E}(w_iw_{i+1}) = \mathcal{E}(w_1w_n) = \dfrac{101}{15}, 1 \leq i \leq n-1.$

Case 6: Let $n \geq 3$. Let $e = w_1u_1$.

Shortest path	$\sigma_e(x,y)$	$\sigma(x,y)$	$\mathcal{E}(e)$
$w_1-u_1, w_n-u_1, u_1-u_n, u_1-u_{n-1}$	1	1	4
$w-u_1$	1	2	$\frac{1}{2}$
$w_{n-1}-u_1$	2	3	$\frac{2}{3}$
$u_1-w_j, 5\le j\le n-2$	1	2	$\frac{1}{2}\times(n-6)$
u_1-u_4	2	5	$\frac{2}{5}$
u_1-u_{n-2}	3	5	$\frac{3}{5}$
$u_1-u_j, 5\le j\le n-3$	2	4	$\frac{2}{4}\times(n-7)$
u_1-w_4	1	3	$\frac{1}{3}$

Hence, $\mathcal{E}(w_iu_i)=\mathcal{E}(u_jw_{j+1})=\mathcal{E}(u_nw_1)=n, 1\le i\le n, 1\le j\le n-1$. □

Definition 2.5. *The triangulated wheel graph TW_{n+1} is obtained by combining a wheel graph W_{n+1} with $V(W_{n+1})=\{w,w_1,w_2,\ldots,w_n\}$ with a set of new vertices $\{u_1,u_2,\ldots,u_n\}$ such that a vertex $u_i\sim w_i,w_{i+1},w$ for $1\le i\le n-1$ and the vertex $u_n\sim w_1,w_n,w$. Note that the diameter of TW_{n+1} is 2.*

Theorem 2.6. *Let $G=TW_{n+1}$, and $V(G)=\{w,w_1,\ldots,w_n,u_1,u_2,\ldots,u_n\}$. Then,*

$$if\ n=3,\quad \mathcal{E}(e)=\begin{cases}\frac{4}{3} & ;e=ww_i, 1\le i\le 3\\ \frac{7}{3} & ;e=wu_i, 1\le i\le 3\\ \frac{5}{3} & ;e=w_iw_{i+1}, 1\le i\le 2,\ and\ w_1w_3\\ \frac{11}{6} & ;e=w_iu_i, w_{i+1}u_i, 1\le i\le 2\ and\ w_3u_3, u_3w_1\end{cases}$$

$$if\ n=4,\quad \mathcal{E}(e)=\begin{cases}\frac{7}{3} & ;e=ww_i, 1\le i\le 4\\ 4 & ;e=wu_i, 1\le i\le 4\\ \frac{8}{3} & ;e=w_iw_{i+1}, w_1w_4, 1\le i\le 3\\ 2 & ;e=w_iu_i, w_{i+1}u_i, 1\le i\le 3\ and\ w_4u_4, u_4w_1\end{cases}$$

$$if\ n\ge 5,\quad \mathcal{E}(e)=\begin{cases}2(n-3) & ;e=ww_i, 1\le i\le n\\ 2(n-2) & ;e=wu_i, 1\le i\le n\\ 3 & ;e=w_iw_{i+1}, 1\le i\le n-1,\ and\ w_1w_n\\ 2 & ;e=w_iu_i, w_{i+1}u_i, 1\le i\le n-1\ and\ w_nu_n, u_nw_1\end{cases}.$$

Proof. Case 1: Let $n=3$. Let $e=ww_1$. Then,

$$\mathcal{E}(e)=\frac{\sigma_e(w,w_1)}{\sigma(w,w_1)}+\frac{\sigma_e(w_1,u_2)}{\sigma(w_1,u_2)}=1+\frac{1}{3}=\frac{4}{3}.$$

Due to symmetry, $\mathcal{E}(ww_i)=\dfrac{4}{3}, 1\le i\le 3$.

Let $e=wu_1$. Then,

$$\mathcal{E}(e)=\frac{\sigma_e(w,u_1)}{\sigma(w,u_1)}+\frac{\sigma_e(u_1,u_3)}{\sigma(u_1,u_3)}+\frac{\sigma_e(w_3,u_1)}{\sigma(w_3,u_1)}+\frac{\sigma_e(u_1,u_2)}{\sigma(u_1,u_2)}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{2}=\frac{7}{3}.$$

Due to symmetry, $\mathcal{E}(wu_i) = \dfrac{7}{3}, 1 \leq i \leq 3$.
Let $e = w_1w_2$. Then,

$$\mathcal{E}(e) = \frac{\sigma_e(w_1,w_2)}{\sigma(w_1,w_2)} + \frac{\sigma_e(w_1,u_2)}{\sigma(w_1,u_2)} + \frac{\sigma_e(w_2,u_3)}{\sigma(w_2,u_3)} = 1 + \frac{1}{3} + \frac{1}{3} = \frac{5}{3}.$$

Due to symmetry, $\mathcal{E}(w_iw_{i+1}) = \mathcal{E}(w_1w_3) = \dfrac{5}{3}, 1 \leq i \leq 2$.

Let $e = w_1u_1$. Then,

$$\mathcal{E}(e) = \frac{\sigma_e(w_1,u_1)}{\sigma(w_1,u_1)} + \frac{\sigma_e(u_1,u_3)}{\sigma(u_1,u_3)} + \frac{\sigma_e(u_1,w_3)}{\sigma(u_1,w_3)} = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}.$$

Due to symmetry, $\mathcal{E}(w_iu_i) = \mathcal{E}(w_{i+1}u_i) = \mathcal{E}(w_3u_3) = \mathcal{E}(u_3w_1) = \dfrac{11}{6}, 1 \leq i \leq 2$.
Case 2: Let $n = 4$. Let $e = ww_1$. Then,

$$\mathcal{E}(e) = \frac{\sigma_e(w,w_1)}{\sigma(w,w_1)} + \frac{\sigma_e(w_1,w_3)}{\sigma(w_1,w_3)} + \frac{\sigma_e(w_1,u_2)}{\sigma(w_1,u_2)} + \frac{\sigma_e(w_1,u_3)}{\sigma(w_1,u_3)} = 1 + \frac{1}{3} + \frac{1}{2} + \frac{1}{2} = \frac{7}{3}.$$

Due to symmetry, $\mathcal{E}(ww_i) = \dfrac{7}{3}, 1 \leq i \leq 4$.
Let $e = wu_1$. Then,

$$\begin{aligned}\mathcal{E}(e) &= \frac{\sigma_e(w,u_1)}{\sigma(w,u_1)} + \frac{\sigma_e(u_1,u_3)}{\sigma(u_1,u_3)} + \frac{\sigma_e(u_1,u_2)}{\sigma(u_1,u_2)} + \frac{\sigma_e(u_1,u_4)}{\sigma(u_1,u_4)} + \frac{\sigma_e(u_1,w_3)}{\sigma(u_1,w_3)} + \frac{\sigma_e(u_1,w_4)}{\sigma(u_1,w_4)} \\ &= 1 + 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 4.\end{aligned}$$

Due to symmetry, $\mathcal{E}(wu_i) = 4, 1 \leq i \leq 4$.

Let w_1w_2. Then,

$$\begin{aligned}\mathcal{E}(e) &= \frac{\sigma_e(w_1,w_2)}{\sigma(w_1,w_2)} + \frac{\sigma_e(w_1,w_3)}{\sigma(w_1,w_3)} + \frac{\sigma_e(w_1,u_2)}{\sigma(w_1,u_2)} + \frac{\sigma_e(w_2,w_4)}{\sigma(w_2,w_4)} + \frac{\sigma_e(w_2,u_4)}{\sigma(w_2,u_4)} \\ &= 1 + \frac{1}{3} + \frac{1}{2} + \frac{1}{3} + \frac{1}{2} = \frac{8}{3}.\end{aligned}$$

Due to symmetry, $\mathcal{E}(w_iw_{i+1}) = \mathcal{E}(w_1w_4) = \dfrac{8}{3}, 1 \leq i \leq 3$.
Let $e = w_1u_1$. Then,

$$\mathcal{E}(e) = \frac{\sigma_e(w_1,u_1)}{\sigma(w_1,u_1)} + \frac{\sigma_e(u_1,u_4)}{\sigma(u_1,u_4)} + \frac{\sigma_e(u_1,w_4)}{\sigma(u_1,w_4)} = 1 + \frac{1}{2} + \frac{1}{2} = 2.$$

Due to symmetry $\mathcal{E}(w_iu_i) = \mathcal{E}(w_{i+1}u_i) = \mathcal{E}(w_4u_4) = \mathcal{E}(u_4w_1) = 2, 1 \leq i \leq 3$.

Case 3: Let $n \geq 5$. Let $e = ww_1$.

Shortest path	$\sigma_e(x,y)$	$\sigma(x,y)$	$\mathcal{E}(e)$
$w - w_1$	1	1	1
$w_1 - w_j, 4 \leq j \leq n-2$	1	1	$n-5$
$w_1 - u_j, 3 \leq j \leq n-2$	1	1	$n-4$
$w_1 - w_3, w_1 - w_{n-1}, w_1 - u_2, w_1 - u_{n-1}$	1	2	$4 \times \frac{1}{2}$

Hence, $\mathcal{E}(ww_i) = 2(n-3), 1 \leq i \leq n$.

Let $e = w_1w_2$.

Shortest path	$\sigma_e(x,y)$	$\sigma(x,y)$	$\mathcal{E}(e)$
$w_1 - w_2$	1	1	1
$w_1 - w_3, w_1 - u_2, w_2 - w_n, w_2 - u_n$	1	2	$4 \times \frac{1}{2}$

Hence, $\mathcal{E}(w_iw_{i+1}) = \mathcal{E}(w_1w_n) = 3, 1 \leq i \leq n-1$.

Let $e = w_1u_1$. Observe that

$$\begin{aligned}\sigma_e(w_1,u_1) &= \sigma(w_1,u_1) = 1,\\ \sigma_e(u_1,u_n) &= \sigma_e(u_1,w_n) = 1,\\ \sigma(u_1,u_n) &= \sigma(u_1,w_n) = 2.\end{aligned}$$

Hence, $\mathcal{E}(w_iu_i) = \mathcal{E}(w_{i+1}u_i) = \mathcal{E}(u_nw_n) = \mathcal{E}(u_nw_1) = 2, 1 \leq i \leq n-1$.

Let $e = wu_1$.

Shortest path	$\sigma_e(x,y)$	$\sigma(x,y)$	$\mathcal{E}(e)$
$w - u_1$	1	1	1
$u_1 - u_j, 3 \leq j \leq n-1$	1	1	$n-3$
$u_1 - u_j, j = 2, n$	1	2	$2 \times \frac{1}{2}$
$u_1 - w_j, 4 \leq j \leq n-1$	1	1	$n-4$
$u_1 - w_j, j = 3, n$	1	2	$2 \times \frac{1}{2}$

Hence, $\mathcal{E}(wu_i) = 2(n-2), 1 \leq i \leq n$. □

Definition 2.6. *The closed flower graph, CF_n is formed by combining a cycle C_n with $V(C_n) = \{w_1, w_2, w_3, \ldots, w_n\}$ and a star graph $K_{1,n}$ with $V(K_{1,n}) = \{u_0, u_1, u_2, u_3, \ldots, u_n\}$, such that vertex $u_i \sim w_i$, $u_i \sim w_{i+1}$ with $1 \leq i \leq n-1$ and $u_n \sim w_1$, $u_n \sim w_n$. The diameter of CF_n is 4.*

Theorem 2.7. *Let $G = CF_n$, and $V(G) = \{w_1, \ldots, w_n, u_0, u_1, \ldots, u_n\}$. Then,*

$$if\ n=3,\quad \mathcal{E}(e) = \begin{cases} 3 & ; e = u_0u_i, 1 \leq i \leq 3\\ 2 & ; e = w_iw_{i+1}, w_1w_3, 1 \leq i \leq 2\\ \frac{5}{2} & ; e = w_iu_i, u_iw_{i+1}, u_3w_3, u_3w_1, 1 \leq i \leq 2 \end{cases},$$

$$if\ n=4,\quad \mathcal{E}(e) = \begin{cases} 4 & ; e = u_0u_i, 1 \leq i \leq 4\\ 4 & ; e = w_iw_{i+1}, w_1w_4, 1 \leq i \leq 3\\ 3 & ; e = w_iu_i, u_iw_{i+1}, u_4w_4, u_4w_1, 1 \leq i \leq 3 \end{cases},$$

$$if\ n=5,\quad \mathcal{E}(e) = \begin{cases} 6 & ; e = u_0u_i, 1 \leq i \leq 5\\ 6 & ; e = w_iw_{i+1}, w_1w_5, 1 \leq i \leq 4\\ \frac{7}{2} & ; e = w_iu_i, u_iw_{i+1}, u_5w_5, u_5w_1, 1 \leq i \leq 4 \end{cases},$$

$$if\ n=6,\quad \mathcal{E}(e)=\begin{cases}\frac{26}{3} & ; e=u_0u_i, 1\le i\le 6\\ \frac{47}{6} & ; e=w_iw_{i+1}, w_1w_6, 1\le i\le 5\\ 4 & ; e=w_iu_i, u_iw_{i+1}, u_6w_6, u_6w_1, 1\le i\le 5\end{cases},$$

$$if\ n=7,\quad \mathcal{E}(e)=\begin{cases}\frac{35}{3} & ; e=u_0u_i, 1\le i\le 7\\ \frac{28}{3} & ; e=w_iw_{i+1}, w_1w_7, 1\le i\le 6\\ \frac{9}{2} & ; e=w_iu_i, u_iw_{i+1}, u_7w_7, u_7w_1, 1\le i\le 6\end{cases},$$

$$if\ n=8,\quad \mathcal{E}(e)=\begin{cases}\frac{46}{3} & ; e=u_0u_i, 1\le i\le 8\\ 10 & ; e=w_iw_{i+1}, w_1w_8, 1\le i\le 7\\ \frac{16}{3} & ; e=w_iu_i, u_iw_{i+1}, u_8w_8, u_8w_1, 1\le i\le 7\end{cases},$$

$$if\ n\ge 9,\quad \mathcal{E}(e)=\begin{cases}\frac{152}{15} & ; e=w_iw_{i+1}, w_nw_1, 1\le i\le n-1\\ \frac{10n-27}{10} & ; e=w_iu_i, u_iw_{i+1}, u_nw_n, u_nw_1, 1\le i\le n-1\\ \frac{60n-251}{15} & ; e=u_0u_i, 1\le i\le n\end{cases}.$$

Proof. Case 1: Let $n=3$. Let $e=u_0u_1$. Then,

$$\begin{aligned}\mathcal{E}(e)&=\frac{\sigma_e(u_0,u_1)}{\sigma(u_0,u_1)}+\sum_{i=1}^{2}\frac{\sigma_e(u_0,w_i)}{\sigma(u_0,w_i)}+\sum_{i=2}^{3}\frac{\sigma_e(u_1,u_i)}{\sigma(u_1,u_i)}\\&=1+2\times\frac{1}{2}+2\times\frac{1}{2}=3.\end{aligned}$$

Hence, $\mathcal{E}(u_0u_i)=3, 1\le i\le 3$.

Let $e=w_1w_2$. Then,

$$\mathcal{E}(e)=\frac{\sigma_e(w_1,w_2)}{\sigma(w_1,w_2)}+\frac{\sigma_e(w_2,u_3)}{\sigma(w_2,u_3)}+\frac{\sigma_e(w_1,u_2)}{\sigma(w_1,u_2)}=1+\frac{1}{2}+\frac{1}{2}=2.$$

Hence, $\mathcal{E}(w_iw_{i+1})=\mathcal{E}(w_1w_3)=2, 1\le i\le 2$.

Let $e=w_1u_1$. Then,

$$\mathcal{E}(e)=\frac{\sigma_e(w_1,u_1)}{\sigma(w_1,u_1)}+\frac{\sigma_e(w_1,u_0)}{\sigma(w_1,u_0)}+\frac{\sigma_e(u_1,u_3)}{\sigma(u_1,u_3)}+\frac{\sigma_e(u_1,w_3)}{\sigma(u_1,w_3)}=1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}=\frac{5}{2}.$$

Hence, $\mathcal{E}(u_iw_i)=\mathcal{E}(u_iw_{i+1})=\mathcal{E}(u_3w_3)=\mathcal{E}(u_3w_1)=\frac{5}{2}, 1\le i\le 2$.

Case 2: Let $n=4$. Let $e=u_0u_1$. Then,

$$\begin{aligned}\mathcal{E}(e)&=\frac{\sigma_e(u_0,u_1)}{\sigma(u_0,u_1)}+\sum_{j=1}^{2}\frac{\sigma_e(u_0,w_j)}{\sigma(u_0,w_j)}+\frac{\sigma_e(u_1,u_2)}{\sigma(u_1,u_2)}+\frac{\sigma_e(u_1,u_4)}{\sigma(u_1,u_4)}+\frac{\sigma_e(u_1,u_3)}{\sigma(u_1,u_3)}\\&=1+\frac{1}{2}\times 2+\frac{1}{2}+\frac{1}{2}+1=4.\end{aligned}$$

Hence, $\mathcal{E}(u_0u_i)=4, 1\le i\le 4$.

Let $e=w_1w_2$. Then,

$$\begin{aligned}\mathcal{E}(e)&=\frac{\sigma_e(w_1,w_2)}{\sigma(w_1,w_2)}+\frac{\sigma_e(w_1,w_3)}{\sigma(w_1,w_3)}+\frac{\sigma_e(w_1,u_2)}{\sigma(w_1,u_2)}+\frac{\sigma_e(w_2,u_4)}{\sigma(w_2,u_4)}+\frac{\sigma_e(w_2,w_4)}{\sigma(w_2,w_4)}\\&=1+\frac{1}{2}+1+1+\frac{1}{2}=4.\end{aligned}$$

Hence, $\mathcal{E}(w_iw_{i+1}) = \mathcal{E}(w_4w_1) = 4, 1 \leq i \leq 3$.

Let $e = w_1u_1$. Then,

$$\begin{aligned}\mathcal{E}(e) &= \frac{\sigma_e(w_1,u_1)}{\sigma(w_1,u_1)} + \frac{\sigma_e(u_1,u_4)}{\sigma(u_1,u_4)} + \frac{\sigma_e(u_1,w_4)}{\sigma(u_1,w_4)} + \frac{\sigma_e(u_0,w_1)}{\sigma(u_0,w_1)}\\ &= 1 + \frac{1}{2} + 1 + \frac{1}{2} = 3.\end{aligned}$$

Hence, $\mathcal{E}(w_iu_i) = \mathcal{E}(u_iw_{i+1}) = \mathcal{E}(w_4u_4) = \mathcal{E}(u_4w_1) = 3, 1 \leq i \leq 3$.

Case 3: Let $n = 5$. Let $e = u_0u_1$. Then,

$$\begin{aligned}\mathcal{E}(e) &= \frac{\sigma_e(u_1,u_0)}{\sigma(u_1,u_0)} + \sum_{j=1}^{2}\frac{\sigma_e(u_0,w_j)}{\sigma(u_0,w_j)} + \sum_{j=3}^{4}\frac{\sigma_e(u_1,u_j)}{\sigma(u_1,u_j)} + \frac{\sigma_e(u_1,u_5)}{\sigma(u_1,u_5)} + \frac{\sigma_e(u_1,u_2)}{\sigma(u_1,u_2)}\\ &+ \frac{\sigma_e(u_1,w_4)}{\sigma(u_1,w_4)} + \frac{\sigma_e(w_2,u_4)}{\sigma(w_1,u_4)} + \frac{\sigma_e(w_1,u_3)}{\sigma(w_1,u_3)}\\ &= 1 + \frac{1}{2}\times 2 + 2 + \frac{1}{2} + \frac{1}{2} + \frac{2}{4} + \frac{1}{4} + \frac{1}{4} = 6.\end{aligned}$$

Hence, $\mathcal{E}(u_0u_i) = 6, 1 \leq i \leq 5$.

Let $e = w_1w_2$. Then,

$$\begin{aligned}\mathcal{E}(e) &= \sum_{j=2}^{3}\frac{\sigma_e(w_1,w_j)}{\sigma(w_1,w_j)} + \frac{\sigma_e(w_1,u_2)}{\sigma(w_1,u_2)} + \frac{\sigma_e(w_2,u_5)}{\sigma(w_2,u_5)} + \frac{\sigma_e(w_2,w_5)}{\sigma(w_2,w_5)} + \frac{\sigma_e(w_1,u_3)}{\sigma(w_1,u_3)}\\ &+ \frac{\sigma_e(w_2,u_4)}{\sigma(w_2,u_4)} + \frac{\sigma_e(w_5,u_2)}{\sigma(w_5,u_2)} + \frac{\sigma_e(w_3,u_5)}{\sigma(w_3,u_5)},\\ &= 2 + 1 + 1 + 1 + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 6.\end{aligned}$$

Due to symmetry, $\mathcal{E}(w_iw_{i+1}) = \mathcal{E}(w_5w_1) = 6, 1 \leq i \leq 4$.

Let $e = w_1u_1$. Then,

$$\begin{aligned}\mathcal{E}(e) &= \frac{\sigma_e(w_1,u_1)}{\sigma(w_1,u_1)} + \frac{\sigma_e(w_1,u_0)}{\sigma(w_1,u_0)} + \frac{\sigma_e(u_1,u_5)}{\sigma(u_1,u_5)} + \frac{\sigma_e(u_1,w_5)}{\sigma(u_1,w_5)} + \frac{\sigma_e(u_1,w_4)}{\sigma(u_1,w_4)} + \frac{\sigma_e(w_1,u_3)}{\sigma(w_1,u_3)}\\ &= 1 + \frac{1}{2} + \frac{1}{2} + 1 + \frac{1}{4} + \frac{1}{4} = \frac{7}{2}.\end{aligned}$$

Hence, $\mathcal{E}(w_iu_i) = \mathcal{E}(u_iw_{i+1}) = \mathcal{E}(u_5w_5) = \mathcal{E}(u_5w_1) = \frac{7}{2}, 1 \leq i \leq 4$.

Case 4: Let $n = 6$. Let $e = u_0u_1$. Then,

$$\begin{aligned}\mathcal{E}(e) &= \frac{\sigma_e(u_0,u_1)}{\sigma(u_0,u_1)} + \sum_{j=1}^{2}\frac{\sigma_e(u_0,w_j)}{\sigma(u_0,w_j)} + \sum_{j=3}^{5}\frac{\sigma_e(u_1,u_j)}{\sigma(u_1,u_j)} + \frac{\sigma_e(u_1,u_6)}{\sigma(u_1,u_6)} + \frac{\sigma_e(u_1,u_2)}{\sigma(u_1,u_2)}\\ &+ \sum_{j=4}^{5}\frac{\sigma_e(u_1,w_j)}{\sigma(u_1,w_j)} + \sum_{j=3}^{4}\frac{\sigma_e(w_1,u_j)}{\sigma(w_1,u_j)} + \sum_{j=4}^{5}\frac{\sigma_e(w_2,u_j)}{\sigma(w_2,u_j)}\\ &= 1 + \frac{1}{2}\times 2 + 3 + \frac{1}{2} + \frac{1}{2} + \frac{2}{3}\times 2 + \frac{1}{3}\times 2 + \frac{1}{3}\times 2 = \frac{26}{3}.\end{aligned}$$

Hence, $\mathcal{E}(u_0u_i) = \dfrac{26}{3}, 1 \leq i \leq 6.$

Let $e = w_1w_2$. Then,

$$\begin{aligned}\mathcal{E}(e) &= \sum_{j=2}^{3} \frac{\sigma_e(w_1,w_j)}{\sigma(w_1,w_j)} + \frac{\sigma_e(w_1,w_4)}{\sigma(w_1,w_4)} + \frac{\sigma_e(w_2,w_6)}{\sigma(w_2,w_6)} + \frac{\sigma_e(w_2,w_5)}{\sigma(w_2,w_5)} + \frac{\sigma_e(w_1,u_2)}{\sigma(w_1,u_2)} \\ &+ \frac{\sigma_e(w_2,u_6)}{\sigma(w_2,u_6)} + \frac{\sigma_e(w_2,u_5)}{\sigma(w_2,u_5)} + \frac{\sigma_e(w_3,u_6)}{\sigma(w_3,u_6)} + \frac{\sigma_e(w_6,u_2)}{\sigma(w_6,u_2)} + \frac{\sigma_e(w_1,u_3)}{\sigma(w_1,u_3)} + \frac{\sigma_e(w_3,w_6)}{\sigma(w_3,w_6)} \\ &= 2 + \frac{1}{2} + 1 + \frac{1}{2} + 1 + 1 + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{2} = \frac{47}{6}.\end{aligned}$$

Due to symmetry, $\mathcal{E}(w_iw_{i+1}) = \mathcal{E}(w_6w_1) = \dfrac{47}{6}, 1 \leq i \leq 5.$

Let $e = w_1u_1$. Then,

$$\begin{aligned}\mathcal{E}(e) &= \frac{\sigma_e(w_1,u_1)}{\sigma(w_1,u_1)} + \frac{\sigma_e(u_1,u_6)}{\sigma(u_1,u_6)} + \frac{\sigma_e(u_1,w_6)}{\sigma(u_1,w_6)} + \frac{\sigma_e(w_1,u_0)}{\sigma(w_1,u_0)} + \sum_{j=3}^{4} \frac{\sigma_e(w_1,u_j)}{\sigma(w_1,u_j)} + \frac{\sigma_e(w_5,u_1)}{\sigma(w_5,u_1)} \\ &= 1 + \frac{1}{2} + 1 + \frac{1}{2} + \frac{1}{3} \times 2 + \frac{1}{3} = 4.\end{aligned}$$

Hence, $\mathcal{E}(w_iu_i) = \mathcal{E}(u_iw_{i+1}) = \mathcal{E}(u_6w_6) = \mathcal{E}(u_6w_1) = 4, 1 \leq i \leq 5.$

Case 5: Let $n = 7$. Let $e = u_0u_1$. Then,

$$\begin{aligned}\mathcal{E}(e) &= \frac{\sigma_e(u_1,u_0)}{\sigma(u_1,u_0)} + \sum_{j=3}^{6} \frac{\sigma_e(u_1,u_j)}{\sigma(u_1,u_j)} + \frac{\sigma_e(u_1,u_2)}{\sigma(u_1,u_2)} + \frac{\sigma_e(u_1,u_7)}{\sigma(u_1,u_7)} + \frac{\sigma_e(u_1,w_4)}{\sigma(u_1,w_4)} + \frac{\sigma_e(u_1,w_5)}{\sigma(u_1,w_5)} \\ &+ \frac{\sigma_e(u_1,w_6)}{\sigma(u_1,w_6)} + \sum_{j=1}^{2} \frac{\sigma_e(u_0,w_j)}{\sigma(u_0,w_j)} + \frac{\sigma_e(w_2,u_6)}{\sigma(w_2,u_6)} + \frac{\sigma_e(w_1,u_5)}{\sigma(w_1,u_5)} + \frac{\sigma_e(w_1,u_4)}{\sigma(w_1,u_4)} + \frac{\sigma_e(w_1,u_3)}{\sigma(w_1,u_3)} \\ &+ \frac{\sigma_e(w_2,u_5)}{\sigma(w_2,u_5)} + \frac{\sigma_e(w_2,u_4)}{\sigma(w_2,u_4)} \\ &= 1 + 4 + \frac{1}{2} + \frac{1}{2} + \frac{2}{3} + 1 + \frac{2}{3} + \frac{1}{2} \times 2 + \frac{1}{3} + \frac{1}{3} + \frac{1}{2} + \frac{1}{3} + \frac{1}{2} + \frac{1}{3} = \frac{35}{3}.\end{aligned}$$

Hence, $\mathcal{E}(u_0u_i) = \dfrac{35}{3}, 1 \leq i \leq 7.$

Let $e = w_1w_2$. Then,

$$\begin{aligned}\mathcal{E}(e) &= \sum_{j=2}^{4} \frac{\sigma_e(w_1,w_j)}{\sigma(w_1,w_j)} + \sum_{j=6}^{7} \frac{\sigma_e(w_2,w_j)}{\sigma(w_2,w_j)} + \frac{\sigma_e(w_1,u_2)}{\sigma(w_1,u_2)} + \frac{\sigma_e(w_1,u_3)}{\sigma(w_1,u_3)} + \frac{\sigma_e(w_2,u_6)}{\sigma(w_2,u_6)} \\ &+ \frac{\sigma_e(w_2,u_7)}{\sigma(w_2,u_7)} + \frac{\sigma_e(w_3,u_7)}{\sigma(w_3,u_7)} + \frac{\sigma_e(w_3,w_7)}{\sigma(w_3,w_7)} + \frac{\sigma_e(w_7,u_2)}{\sigma(w_7,u_2)} \\ &= 3 + 2 + 1 + \frac{1}{3} + \frac{1}{3} + 1 + \frac{1}{3} + 1 + \frac{1}{3} = \frac{28}{3}.\end{aligned}$$

Hence, $\mathcal{E}(w_iw_{i+1}) = \mathcal{E}(w_7w_1) = \dfrac{28}{3}, 1 \leq i \leq 6.$

Let $e = w_1u_1$. Then,

$$\begin{aligned}\mathcal{E}(e) &= \frac{\sigma_e(w_1,u_1)}{\sigma(w_1,u_1)} + \frac{\sigma_e(w_1,u_0)}{\sigma(w_1,u_0)} + \frac{\sigma_e(w_7,u_1)}{\sigma(w_7,u_1)} + \frac{\sigma_e(w_6,u_1)}{\sigma(w_6,u_1)} + \frac{\sigma_e(u_7,u_1)}{\sigma(u_7,u_1)} \\ &\quad + \frac{\sigma_e(w_1,u_3)}{\sigma(w_1,u_3)} + \frac{\sigma_e(w_1,u_4)}{\sigma(w_1,u_4)} + \frac{\sigma_e(w_1,u_5)}{\sigma(w_1,u_5)} \\ &= 1 + \frac{1}{2} + 1 + \frac{1}{3} + \frac{1}{2} + \frac{1}{3} + \frac{1}{2} + \frac{1}{3} = \frac{9}{2}.\end{aligned}$$

Hence, $\mathcal{E}(w_iu_i) = \mathcal{E}(u_iw_{i+1}) = \mathcal{E}(u_7w_7) = \mathcal{E}(u_7w_1) = \frac{9}{2}, 1 \le i \le 6$.

Case 6: Let $n = 8$. Let $e = u_0u_1$. Then,

$$\begin{aligned}\mathcal{E}(e) &= \sum_{j=3}^{7}\frac{\sigma_e(u_1,u_j)}{\sigma(u_1,u_j)} + \sum_{j=5}^{6}\frac{\sigma_e(u_1,w_j)}{\sigma(u_1,w_j)} + \sum_{j=1}^{2}\frac{\sigma_e(u_0,w_j)}{\sigma(u_0,w_j)} + \sum_{j=5}^{6}\frac{\sigma_e(w_2,u_j)}{\sigma(w_2,u_j)} + \sum_{j=4}^{5}\frac{\sigma_e(w_1,u_j)}{\sigma(w_1,u_j)} \\ &\quad + \frac{\sigma_e(u_1,u_2)}{\sigma(u_1,u_2)} + \frac{\sigma_e(u_1,u_8)}{\sigma(u_1,u_8)} + \frac{\sigma_e(u_1,w_4)}{\sigma(u_1,w_4)} + \frac{\sigma_e(u_1,w_7)}{\sigma(u_1,w_7)} + \frac{\sigma_e(u_7,w_2)}{\sigma(u_7,w_2)} + \frac{\sigma_e(w_2,u_4)}{\sigma(w_2,u_4)} \\ &\quad + \frac{\sigma_e(w_2,w_6)}{\sigma(w_2,w_6)} + \frac{\sigma_e(w_1,w_5)}{\sigma(w_1,w_5)} + \frac{\sigma_e(w_1,u_3)}{\sigma(w_1,u_3)} + \frac{\sigma_e(w_1,u_6)}{\sigma(w_1,u_6)} + \frac{\sigma_e(u_0,u_1)}{\sigma(u_0,u_1)} \\ &= 5 + 2 + \frac{1}{2}\times 2 + \frac{1}{2}\times 2 + \frac{1}{2}\times 2 + \frac{1}{2} + \frac{1}{2} + \frac{2}{3} + \frac{2}{3} + \frac{1}{3} + \frac{1}{3} + \frac{2}{6} + \frac{2}{6} + \frac{1}{3} + \frac{1}{3} + 1 \\ &= \frac{46}{3}.\end{aligned}$$

Hence, $\mathcal{E}(u_0u_i) = \frac{46}{3}, 1 \le i \le 8$.

Let $e = w_1w_2$. Then,

$$\begin{aligned}\mathcal{E}(e) &= \sum_{j=2}^{4}\frac{\sigma_e(w_1,w_j)}{\sigma(w_1,w_j)} + \sum_{j=7}^{8}\frac{\sigma_e(w_2,w_j)}{\sigma(w_2,w_j)} + \frac{\sigma_e(w_1,w_5)}{\sigma(w_1,w_5)} + \frac{\sigma_e(w_2,w_6)}{\sigma(w_2,w_6)} + \frac{\sigma_e(w_2,u_8)}{\sigma(w_2,u_8)} \\ &\quad + \frac{\sigma_e(w_2,u_7)}{\sigma(w_2,u_7)} + \frac{\sigma_e(w_1,u_3)}{\sigma(w_1,u_3)} + \frac{\sigma_e(w_3,w_8)}{\sigma(w_3,w_8)} + \frac{\sigma_e(w_3,u_8)}{\sigma(w_3,u_8)} + \frac{\sigma_e(w_8,u_2)}{\sigma(w_8,u_2)} \\ &\quad + \frac{\sigma_e(w_3,w_7)}{\sigma(w_3,w_7)} + \frac{\sigma_e(w_8,w_4)}{\sigma(w_8,w_4)} + \frac{\sigma_e(w_1,u_2)}{\sigma(w_1,u_2)} \\ &= 3 + 2 + \frac{1}{6} + \frac{1}{6} + 1 + \frac{1}{3} + \frac{1}{3} + 1 + \frac{1}{3} + \frac{1}{3} + \frac{1}{6} + \frac{1}{6} + 1 = 10.\end{aligned}$$

Hence, $\mathcal{E}(w_iw_{i+1}) = \mathcal{E}(w_8w_1) = 10, 1 \le i \le 7$.

Let $e = w_1u_1$. Then,

$$\begin{aligned}\mathcal{E}(e) &= \frac{\sigma_e(w_1,u_1)}{\sigma(w_1,u_1)} + \frac{\sigma_e(w_1,u_0)}{\sigma(w_1,u_0)} + \frac{\sigma_e(w_1,u_3)}{\sigma(w_1,u_3)} + \sum_{j=4}^{5}\frac{\sigma_e(w_1,u_j)}{\sigma(w_1,u_j)} + \frac{\sigma_e(w_1,u_6)}{\sigma(w_1,u_6)} \\ &\quad + \frac{\sigma_e(u_1,u_8)}{\sigma(u_1,u_8)} + \frac{\sigma_e(u_1,w_8)}{\sigma(u_1,w_8)} + \frac{\sigma_e(u_1,w_7)}{\sigma(u_1,w_7)} + \frac{\sigma_e(w_1,w_5)}{\sigma(w_1,w_5)} \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{2}\times 2 + \frac{1}{3} + \frac{1}{2} + 1 + \frac{1}{3} + \frac{2}{6} = \frac{16}{3}.\end{aligned}$$

Hence, $\mathcal{E}(w_iu_i) = \mathcal{E}(u_iw_{i+1}) = \mathcal{E}(u_8w_8) = \mathcal{E}(u_8w_1) = \frac{16}{3}, 1 \le i \le 7$.

Case 7: Let $n \geq 9$. Let $e = u_0u_1$.

Shortest path	$\sigma_e(x,y)$	$\sigma(x,y)$	$\mathcal{E}(e)$
$u_0 - u_1$	1	1	1
$u_0 - w_1, u_0 - w_2, u_1 - u_2, u_1 - u_n$	1	2	$\frac{1}{2} \times 4$
$u_1 - u_i, 3 \leq i \leq n-1$	1	1	$n-3$
$u_1 - w_i, 5 \leq i \leq n-2$	2	2	$n-6$
$u_1 - w_4, u_1 - w_{n-1}$	2	3	$\frac{2}{3} \times 2$
$w_2 - u_i, 5 \leq i \leq n-2$	1	2	$\frac{1}{2} \times (n-6)$
$w_1 - u_j, 4 \leq j \leq n-3$	1	2	$\frac{1}{2} \times (n-6)$
$w_2 - u_4, w_2 - u_{n-1}, w_1 - u_3, w_1 - u_{n-2}$	1	3	$\frac{1}{3} \times 4$
$w_2 - w_6, w_2 - w_{n-2}, w_1 - w_5, w_1 - w_{n-3}$	2	5	$\frac{2}{5} \times 4$
$w_2 - w_i, 7 \leq i \leq n-3$	1	2	$\frac{1}{2} \times (n-9)$
$w_1 - w_i, 6 \leq i \leq n-4$	1	2	$\frac{1}{2} \times (n-9)$

Hence, $\mathcal{E}(u_0u_i) = \frac{60n-251}{15}, 1 \leq i \leq n.$

Let $e = w_1w_2$.

Shortest path	$\sigma_e(x,y)$	$\sigma(x,y)$	$\mathcal{E}(e)$
$w_1 - w_i, 2 \leq i \leq 4$	1	1	1×3
$w_1 - u_3, w_n - u_2, w_2 - u_{n-1}, w_3 - u_n$	1	3	$\frac{1}{3} \times 4$
$w_1 - w_5, w_2 - w_{n-2}, w_3 - w_{n-1}, w_4 - w_n$	1	5	$\frac{1}{5} \times 4$
$w_2 - w_i, i = n, n-1$	1	1	1×2
$w_3 - w_n, w_2 - u_n, w_1 - u_2$	1	1	1×3

Hence, $\mathcal{E}(w_iw_{i+1}) = \mathcal{E}(w_nw_1) = \frac{152}{15}, 1 \leq i \leq n-1.$

Let $e = w_1u_1$.

Shortest path	$\sigma_e(x,y)$	$\sigma(x,y)$	$\mathcal{E}(e)$
$w_1 - u_1, w_n - u_1$	1	1	1×2
$w_1 - u_i, 4 \leq i \leq n-3$	1	2	$\frac{1}{2} \times (n-6)$
$w_1 - w_i, 6 \leq i \leq n-4$	1	2	$\frac{1}{2} \times (n-9)$
$w_1 - w_5, w_1 - w_{n-3}$	2	5	$\frac{2}{5} \times 2$
$w_1 - u_3, w_1 - u_{n-2}, u_1 - w_{n-1}$	1	3	$\frac{1}{3} \times 3$
$w_1 - u_0, u_1 - u_n$	1	2	$\frac{1}{2} \times 2$

Hence, $\mathcal{E}(w_iu_i) = \mathcal{E}(w_{i+1}u_i) = \mathcal{E}(w_nu_n) = \mathcal{E}(w_1u_n) = \frac{10n-27}{10}, 1 \leq i \leq n-1.$ □

Definition 2.7. *The double crown-wheel graph, DCW_n is constructed from the wheel graph W_{n+1} with $V(W_{n+1}) = \{w, w_1, w_2, w_3, \ldots, w_n\}$ by adding two distinct vertex sets.*

1. *The first set is $\{u_1, u_2, u_3, \ldots, u_n\}$, and $u_i \sim w_i$, $u_i \sim w_{i+1}$ for $1 \le i \le n-1$, and $u_n \sim w_1$, $u_n \sim w_n$.*
2. *The second set is $\{v_1, v_2, v_3, \ldots, v_n\}$, and $v_i \sim w_i$, $v_i \sim w_{i+1}$, for $1 \le i \le n-1$, and $v_n \sim w_1$, $v_n \sim w_n$. Note that the diameter of DCW_n is 4.*

Theorem 2.8. *Let $G = DCW_n$, and $V(G) = \{w, w_1, \ldots, w_n, u_1, \ldots, u_n, v_1, \ldots, v_n\}$. Then,*

$$if\ n = 3, \quad \mathcal{E}(e) = \begin{cases} 3 & ; e = ww_i, 1 \le i \le 3 \\ 3 & ; e = w_iw_{i+1}, w_1w_3, 1 \le i \le 2 \end{cases},$$

$$if\ n = 4, \quad \mathcal{E}(e) = \begin{cases} \frac{10}{3} & ; e = ww_i, 1 \le i \le 4 \\ \frac{23}{3} & ; e = w_iw_{i+1}, w_1w_4, 1 \le i \le 3 \end{cases},$$

$$if\ n = 5, \quad \mathcal{E}(e) = \begin{cases} 6 & ; e = ww_i, 1 \le i \le 5 \\ 12 & ; e = w_iw_{i+1}, w_1w_5, 1 \le i \le 4 \end{cases},$$

$$if\ n = 6, \quad \mathcal{E}(e) = \begin{cases} 13 & ; e = ww_i, 1 \le i \le 6 \\ 14 & ; e = w_iw_{i+1}, w_1w_6, 1 \le i \le 5 \end{cases},$$

$$if\ n \ge 7, \quad \mathcal{E}(e) = \begin{cases} \frac{135n-619}{15} & ; e = ww_i, 1 \le i \le n \\ \frac{214}{15} & ; e = w_iw_{i+1}, w_1w_n, 1 \le i \le n-1 \end{cases},$$

$$\text{if } n \ge 3,\ \mathcal{E}(e) = \frac{3n}{2} \text{ if } e = w_iu_i, u_iw_{i+1}, w_iv_i, v_iw_{i+1}, u_nw_n, u_nw_1, v_nw_n, v_nw_1, 1 \le i \le n-1.$$

Proof. Case 1: Let $n = 3$. Let $e = ww_1$. Then,

$$\begin{aligned} \mathcal{E}(e) &= \frac{\sigma_e(w, w_1)}{\sigma(w, w_1)} + \frac{\sigma_e(w, v_1)}{\sigma(w, v_1)} + \frac{\sigma_e(w, v_3)}{\sigma(w, v_3)} + \frac{\sigma_e(w, u_1)}{\sigma(w, u_1)} + \frac{\sigma_e(w, u_3)}{\sigma(w, u_3)} \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 3. \end{aligned}$$

Hence, $\mathcal{E}(ww_i) = 3, 1 \le i \le 3$.

Let $e = w_1w_2$. Then,

$$\begin{aligned} \mathcal{E}(e) &= \frac{\sigma_e(w_1, w_2)}{\sigma(w_1, w_2)} + \frac{\sigma_e(w_2, v_3)}{\sigma(w_2, v_3)} + \frac{\sigma_e(w_2, u_3)}{\sigma(w_2, u_3)} + \frac{\sigma_e(w_1, u_2)}{\sigma(w_1, u_2)} + \frac{\sigma_e(w_1, v_2)}{\sigma(w_1, v_2)} \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 3. \end{aligned}$$

Hence, $\mathcal{E}(w_iw_{i+1}) = \mathcal{E}(w_3w_1) = 3, 1 \le i \le 2$.

Case 2: Let $n = 4$. Let $e = ww_1$. Then,

$$\begin{aligned} \mathcal{E}(e) &= \frac{\sigma_e(w, w_1)}{\sigma(w, w_1)} + \frac{\sigma_e(w_1, w_3)}{\sigma(w_1, w_3)} + \frac{\sigma_e(w, u_1)}{\sigma(w, u_1)} + \frac{\sigma_e(w, v_1)}{\sigma(w, v_1)} + \frac{\sigma_e(w, u_4)}{\sigma(w, u_4)} + \frac{\sigma_e(w, v_4)}{\sigma(w, v_4)} \\ &= 1 + \frac{1}{3} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{10}{3}. \end{aligned}$$

Hence, $\mathcal{E}(ww_i) = \dfrac{10}{3}, 1 \leq i \leq 4.$

Let $e = w_1w_2$. Then,

$$\begin{aligned}\mathcal{E}(e) &= \frac{\sigma_e(w_1,w_2)}{\sigma(w_1,w_2)} + \frac{\sigma_e(w_1,u_2)}{\sigma(w_1,u_2)} + \frac{\sigma_e(w_1,v_2)}{\sigma(w_1,v_2)} + \frac{\sigma_e(w_2,u_4)}{\sigma(w_2,u_4)} + \frac{\sigma_e(w_2,v_4)}{\sigma(w_2,v_4)} + \frac{\sigma_e(w_2,w_4)}{\sigma(w_2,w_4)}\\ &\quad + \frac{\sigma_e(w_1,w_3)}{\sigma(w_1,w_3)} + \frac{\sigma_e(u_2,u_4)}{\sigma(u_2,u_4)} + \frac{\sigma_e(u_4,v_2)}{\sigma(u_4,v_2)} + \frac{\sigma_e(v_4,u_2)}{\sigma(v_4,u_2)} + \frac{\sigma_e(v_2,v_4)}{\sigma(v_2,v_4)}\\ &= 1+1+1+1+1+\frac{1}{3}+\frac{1}{3}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2} = \frac{23}{3}.\end{aligned}$$

Hence, $\mathcal{E}(w_iw_{i+1}) = \mathcal{E}(w_4w_1) = \dfrac{23}{3}, 1 \leq i \leq 3.$

Case 3: Let $n = 5$. Let $e = ww_1$. Then,

$$\begin{aligned}\mathcal{E}(e) &= \frac{\sigma_e(w,w_1)}{\sigma(w,w_1)} + \sum_{j=3}^{4}\frac{\sigma_e(w_1,w_j)}{\sigma(w_1,w_j)} + \frac{\sigma_e(w,u_1)}{\sigma(w,u_1)} + \frac{\sigma_e(w,u_5)}{\sigma(w,u_5)} + \frac{\sigma_e(w,v_1)}{\sigma(w,v_1)} + \frac{\sigma_e(w,v_5)}{\sigma(w,v_5)}\\ &\quad + \frac{\sigma_e(w_3,u_5)}{\sigma(w_3,u_5)} + \frac{\sigma_e(w_3,v_5)}{\sigma(w_3,v_5)} + \frac{\sigma_e(w_4,u_1)}{\sigma(w_4,u_1)} + \frac{\sigma_e(w_4,v_1)}{\sigma(w_4,v_1)} + \frac{\sigma_e(w_1,u_3)}{\sigma(w_1,u_3)} + \frac{\sigma_e(w_1,v_3)}{\sigma(w_1,v_3)}\\ &= 1+\frac{1}{2}\times 2+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{2}{4}+\frac{2}{4} = 6.\end{aligned}$$

Hence, $\mathcal{E}(ww_i) = 6, 1 \leq i \leq 5.$

Let $e = w_1w_2$. Then,

$$\begin{aligned}\mathcal{E}(e) &= \frac{\sigma_e(w_1,w_2)}{\sigma(w_1,w_2)} + \frac{\sigma_e(w_1,w_3)}{\sigma(w_1,w_3)} + \frac{\sigma_e(w_2,w_5)}{\sigma(w_2,w_5)} + \frac{\sigma_e(w_1,u_2)}{\sigma(w_1,u_2)} + \frac{\sigma_e(w_1,v_2)}{\sigma(w_1,v_2)} + \frac{\sigma_e(w_2,u_5)}{\sigma(w_2,u_5)}\\ &\quad + \frac{\sigma_e(w_2,v_5)}{\sigma(w_2,v_5)} + \frac{\sigma_e(w_1,u_3)}{\sigma(w_1,u_3)} + \frac{\sigma_e(w_1,v_3)}{\sigma(w_1,v_3)} + \frac{\sigma_e(w_2,u_4)}{\sigma(w_2,u_4)} + \frac{\sigma_e(w_2,v_4)}{\sigma(w_2,v_4)} + \frac{\sigma_e(u_5,u_2)}{\sigma(u_5,u_2)}\\ &\quad + \frac{\sigma_e(v_5,u_2)}{\sigma(v_5,u_2)} + \frac{\sigma_e(u_5,v_2)}{\sigma(u_5,v_2)} + \frac{\sigma_e(v_5,v_2)}{\sigma(v_5,v_2)} + \frac{\sigma_e(w_5,u_2)}{\sigma(w_5,u_2)} + \frac{\sigma_e(w_5,v_2)}{\sigma(w_5,v_2)} + \frac{\sigma_e(w_3,u_5)}{\sigma(w_3,u_5)}\\ &\quad + \frac{\sigma_e(w_3,v_5)}{\sigma(w_3,v_5)}\\ &= 1+\frac{1}{2}+\frac{1}{2}+1+1+1+1+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+1+1+1+1+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\\ &= 12.\end{aligned}$$

Hence, $\mathcal{E}(w_iw_{i+1}) = \mathcal{E}(w_5w_1) = 12, 1 \leq i \leq 4.$

Case 4: Let $n = 6$. Let $e = ww_1$. Then,

$$\begin{aligned}
\mathcal{E}(e) &= \frac{\sigma_e(w,w_1)}{\sigma(w,w_1)} + \sum_{j=3}^{4} \frac{\sigma_e(w_1,u_j)}{\sigma(w_1,u_j)} + \sum_{j=3}^{4} \frac{\sigma_e(w_1,v_j)}{\sigma(w_1,v_j)} + \sum_{j=3}^{4} \frac{\sigma_e(u_6,w_j)}{\sigma(u_6,w_j)} + \sum_{j=3}^{4} \frac{\sigma_e(v_6,w_j)}{\sigma(v_6,w_j)} \\
&+ \sum_{j=4}^{5} \frac{\sigma_e(u_1,w_j)}{\sigma(u_1,w_j)} + \sum_{j=4}^{5} \frac{\sigma_e(v_1,w_j)}{\sigma(v_1,w_j)} + \frac{\sigma_e(u_3,u_6)}{\sigma(u_3,u_6)} + \frac{\sigma_e(v_3,u_6)}{\sigma(v_3,u_6)} + \frac{\sigma_e(v_6,u_3)}{\sigma(v_6,u_3)} + \frac{\sigma_e(v_6,v_3)}{\sigma(v_6,v_3)} \\
&+ \frac{\sigma_e(u_4,u_1)}{\sigma(u_4,u_1)} + \frac{\sigma_e(u_4,v_1)}{\sigma(u_4,v_1)} + \frac{\sigma_e(v_4,u_1)}{\sigma(v_4,u_1)} + \frac{\sigma_e(v_4,v_1)}{\sigma(v_4,v_1)} + \frac{\sigma_e(w,u_6)}{\sigma(w,u_6)} + \frac{\sigma_e(w,v_6)}{\sigma(w,v_6)} \\
&+ \frac{\sigma_e(w,u_1)}{\sigma(w,u_1)} + \frac{\sigma_e(w,v_1)}{\sigma(w,v_1)} + \frac{\sigma_e(w_1,w_5)}{\sigma(w_1,w_5)} + \frac{\sigma_e(w_1,w_4)}{\sigma(w_1,w_4)} + \frac{\sigma_e(w_1,w_3)}{\sigma(w_1,w_3)}, \\
&= 1 + \frac{2}{3} \times 2 + \frac{2}{3} \times 2 + \frac{1}{3} \times 2 + \frac{1}{3} \times 2 + \frac{1}{3} \times 2 + \frac{1}{3} \times 2 + \frac{2}{6} + \frac{2}{6} + \frac{2}{6} + \frac{2}{6} + \frac{2}{6} \\
&+ \frac{2}{6} + \frac{2}{6} + \frac{2}{6} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + 1 + \frac{1}{2} = 13.
\end{aligned}$$

Hence, $\mathcal{E}(ww_i) = 13, 1 \leq i \leq 5$.

Let $e = w_1w_2$. Then,

$$\begin{aligned}
\mathcal{E}(e) &= \frac{\sigma_e(w_1,w_2)}{\sigma(w_1,w_2)} + \frac{\sigma_e(w_1,w_3)}{\sigma(w_1,w_3)} + \frac{\sigma_e(w_1,u_2)}{\sigma(w_1,u_2)} + \frac{\sigma_e(w_1,v_2)}{\sigma(w_1,v_2)} + \frac{\sigma_e(w_2,u_6)}{\sigma(w_2,u_6)} + \frac{\sigma_e(w_2,v_6)}{\sigma(w_2,v_6)} \\
&+ \frac{\sigma_e(w_2,w_6)}{\sigma(w_2,w_6)} + \frac{\sigma_e(w_1,u_3)}{\sigma(w_1,u_3)} + \frac{\sigma_e(w_1,v_3)}{\sigma(w_1,v_3)} + \frac{\sigma_e(w_2,u_5)}{\sigma(w_2,u_5)} + \frac{\sigma_e(u_2,u_5)}{\sigma(u_2,u_5)} + \frac{\sigma_e(u_2,v_5)}{\sigma(u_2,v_5)} \\
&+ \frac{\sigma_e(v_2,u_5)}{\sigma(v_2,u_5)} + \frac{\sigma_e(v_2,v_5)}{\sigma(v_2,v_5)} + \frac{\sigma_e(u_6,u_3)}{\sigma(u_6,u_3)} + \frac{\sigma_e(u_6,v_3)}{\sigma(u_6,v_3)} + \frac{\sigma_e(v_6,u_3)}{\sigma(v_6,u_3)} + \frac{\sigma_e(v_6,v_3)}{\sigma(v_6,v_3)} \\
&+ \frac{\sigma_e(u_6,u_2)}{\sigma(u_6,u_2)} + \frac{\sigma_e(u_6,v_2)}{\sigma(u_6,v_2)} + \frac{\sigma_e(v_6,u_2)}{\sigma(v_6,u_2)} + \frac{\sigma_e(v_6,v_2)}{\sigma(v_6,v_2)} + \frac{\sigma_e(u_6,w_3)}{\sigma(u_6,w_3)} + \frac{\sigma_e(v_6,w_3)}{\sigma(v_6,w_3)} \\
&+ \frac{\sigma_e(w_6,u_2)}{\sigma(w_6,u_2)} + \frac{\sigma_e(w_6,v_2)}{\sigma(w_6,v_2)} + \frac{\sigma_e(w_2,v_5)}{\sigma(w_2,v_5)} \\
&= 1 + \frac{1}{2} + 1 + 1 + 1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} \\
&+ 1 + 1 + 1 + 1 + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 14.
\end{aligned}$$

Hence, $\mathcal{E}(w_iw_{i+1}) = \mathcal{E}(w_5w_1) = 14, 1 \leq i \leq 4$.

Case 5: Let $n \geq 7$. Let $e = ww_1$.

Shortest path	$\sigma_e(x,y)$	$\sigma(x,y)$	$\mathcal{E}(e)$
$w_1 - w$	1	1	1
$w - u_i, w - v_i, i = 1, n$	1	2	$\frac{1}{2} \times 4$
$w_1 - w_3, w_1 - w_{n-1}$	1	2	$\frac{1}{2} \times 2$
$w_1 - w_i, 4 \leq i \leq n-2$	1	1	$n-5$
$w_1 - u_i, w_1 - v_i, i = 3, n-2$	2	3	$\frac{2}{3} \times 4$
$w_1 - u_i, w_1 - v_i, 4 \leq i \leq n-3$	2	2	$2(n-6)$
$u_1 - u_i, u_1 - v_i, v_1 - u_i, v_1 - v_i, i = 4, n-2$	2	5	$\frac{2}{5} \times 8$
$u_n - u_i, u_1 - v_i, v_n - u_i, v_1 - v_i, i = 3, n-3$	2	5	$\frac{2}{5} \times 8$
$u_1 - u_i, u_1 - v_i, v_1 - u_i, v_1 - v_i, 5 \leq i \leq n-3$	2	4	$\frac{2}{4} \times 4(n-7)$
$u_n - u_i, u_n - v_i, v_n - u_i, v_n - v_i, 4 \leq i \leq n-4$	1	2	$\frac{1}{2} \times 4(n-7)$
$u_1 - w_i, v_1 - w_i, i = 4, n-1$	1	3	$\frac{1}{3} \times 4$
$u_1 - w_i, v_1 - w_i, 5 \leq i \leq n-2$	1	2	$\frac{1}{2} \times 2(n-6)$
$u_n - w_i, v_n - w_i, i = 3, n-2$	1	3	$\frac{1}{3} \times 4$
$u_n - w_i, v_n - w_i, 4 \leq i \leq n-3$	1	2	$\frac{1}{2} \times 2(n-6)$

Hence, $\mathcal{E}(ww_i) = \frac{135n-619}{15}, 1 \leq i \leq n.$

Let $e = w_1w_2$.

Shortest path	$\sigma_e(x,y)$	$\sigma(x,y)$	$\mathcal{E}(e)$
$w_1 - w_2, w_1 - u_2, w_1 - v_2, w_2 - u_n, w_2 - v_n$	1	1	1×5
$w_1 - u_3, w_1 - v_3, w_2 - u_{n-1}, w_2 - v_{n-1}, w_3 - u_n, w_3 - v_n$	1	3	$\frac{1}{3} \times 6$
$u_2 - u_n, u_2 - v_n, v_2 - u_n, v_2 - v_n$	1	1	1×4
$w_1 - w_3, w_2 - w_n$	1	2	$\frac{1}{2} \times 2$
$w_n - u_2, w_n - v_2$	1	3	$\frac{1}{3} \times 2$
$u_3 - u_n, u_3 - v_n, v_3 - u_n, v_3 - v_n, u_2 - u_{n-1}$, $u_2 - v_{n-1}, v_2 - u_{n-1}, v_2 - v_{n-1}$	1	5	$\frac{1}{5} \times 8$

Hence, $\mathcal{E}(w_iw_{i+1}) = \mathcal{E}(w_1w_n) = \frac{214}{15}, 1 \leq i \leq n-1.$

Case 6: Let $n \geq 3$. Let $e = w_1u_1$.

Shortest path	$\sigma_e(x,y)$	$\sigma(x,y)$	$\mathcal{E}(e)$
$u_1-w_1, u_1-u_n, u_1-v_n, u_1-w_n,$ u_1-u_{n-1}, u_1-v_{n-1}	1	1	6
$u_1-w_i, 5\leq i\leq n-2$	1	2	$\frac{1}{2}\times(n-6)$
u_1-w_{n-1}	2	3	$\frac{2}{3}$
u_1-w_4	1	3	$\frac{1}{3}$
u_1-u_4, u_1-v_4	2	5	$\frac{2}{5}\times 2$
u_1-u_{n-2}, u_1-v_{n-2}	3	5	$\frac{3}{5}\times 2$
$u_1-u_i, u_1-v_i, 5\leq i\leq n-3$	2	4	$\frac{2}{4}\times 2(n-7)$
u_1-w, u_1-v_1	1	2	$\frac{1}{2}\times 2$

Hence, $\mathcal{E}(u_iw_i)=\mathcal{E}(u_iw_{i+1})=\mathcal{E}(w_iv_i)=\mathcal{E}(v_iw_{i+1})=\dfrac{3n}{2}, 1\leq i\leq n-1$, and
$\mathcal{E}(u_nw_n)=\mathcal{E}(u_nw_1)=\mathcal{E}(v_nw_n)=\mathcal{E}(v_nw_1)=\dfrac{3n}{2}$. □

3 Conclusions

EBCM of edges in wheel-related graphs is discussed in this article. A similar study can be carried out on different classes of graphs, as well as various derived graphs obtained by different graph operations.

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